Determinacy, Large Cardinals, and Inner Models

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January 2024

Winterschool 2024, Hejnice

Research supported by Austrian Science Fund (FWF) Elise Richter grant number V844, International Project I6087, and START Prize Y1498.

How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.

Determinacy

Large Cardinals

Forcing Axioms

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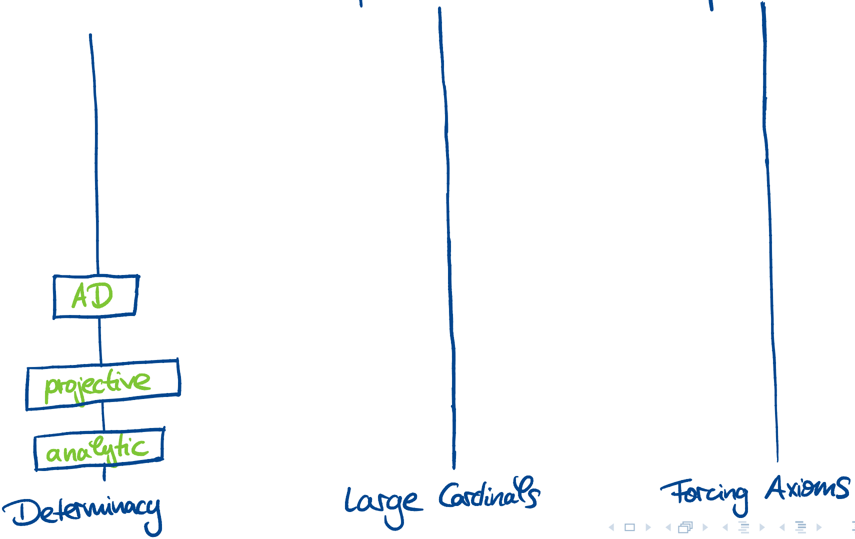

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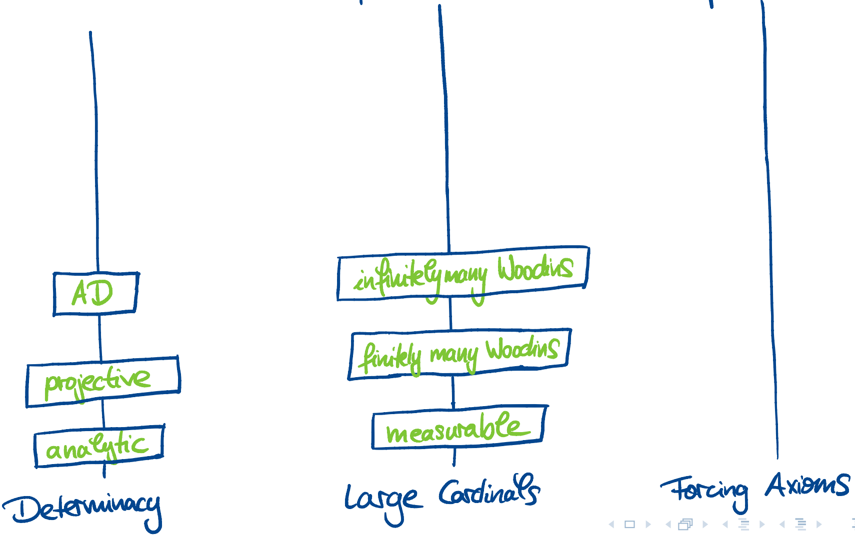

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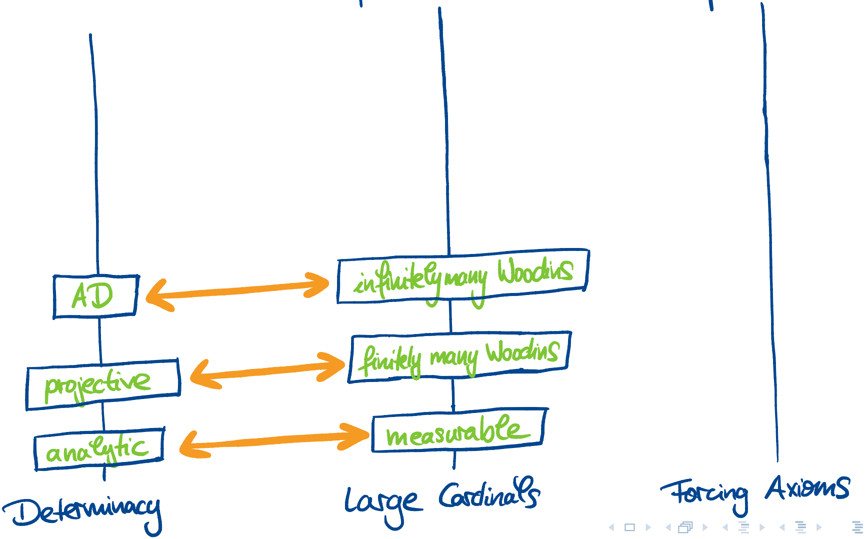

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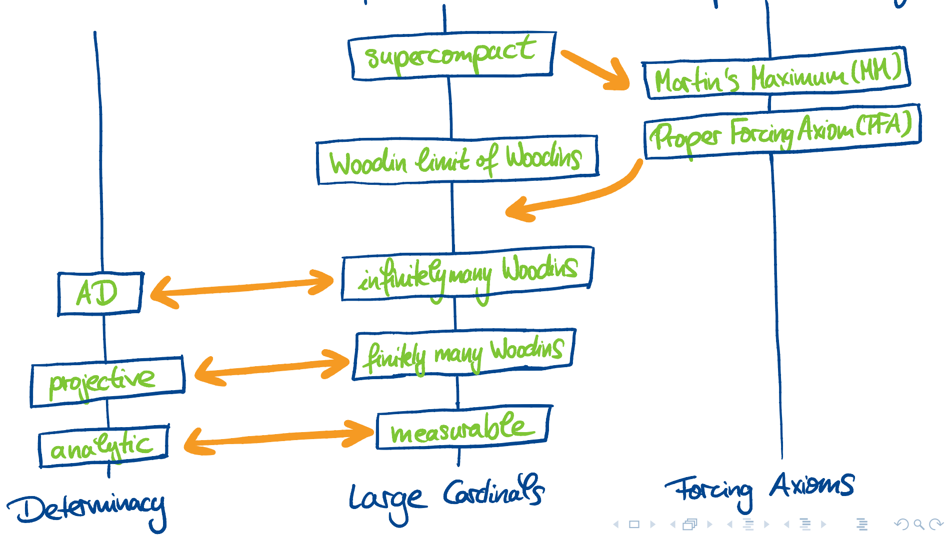

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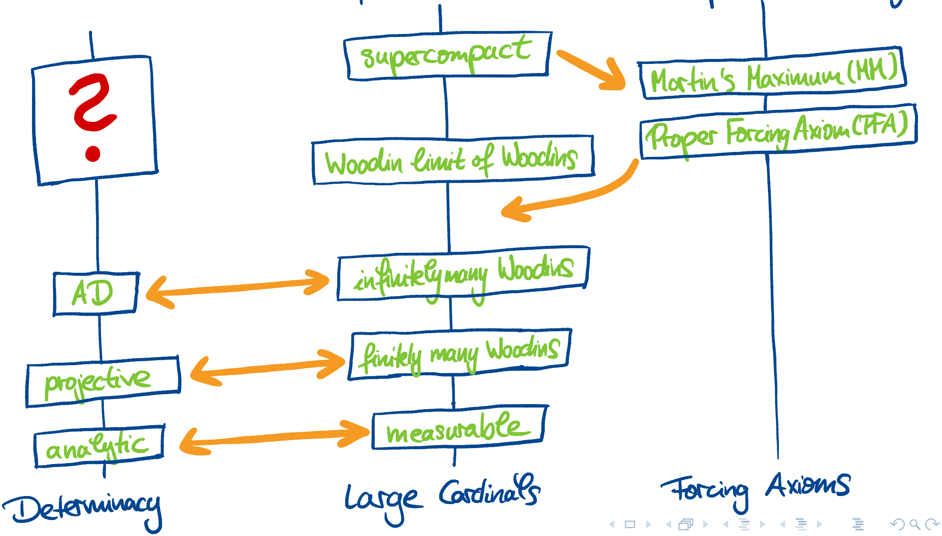

Two scenarios

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$\boxed{\text{Strong models}}$

Another approach to strengthen determinacy

Keep playing games of length ω and impose additional structural properties on the model.

Universally Baire sets: the definition

Definition (Schilling-Vaught, Feng-Magidor-Woodin)

A subset A of a topological space Y is *universally Baire* if for every topological space X and continuous $f\colon X \to Y$,

$$f^{-1}{}''A \text{ has the property of Baire in } X.$$

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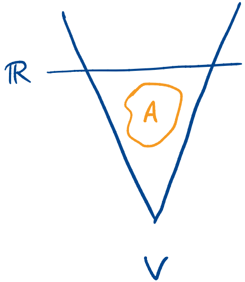
But this is NOT the definition we want to use.

Universally Baire sets: the set-theoretic picture

A set of reals is universally Baire if it can be canonically extended onto generic extensions.



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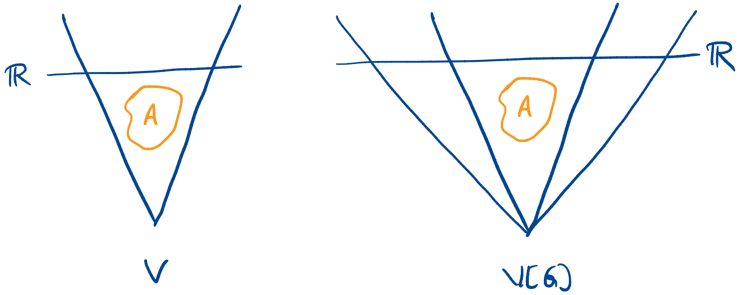

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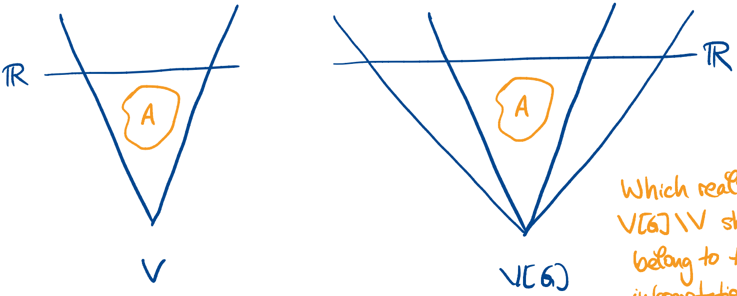

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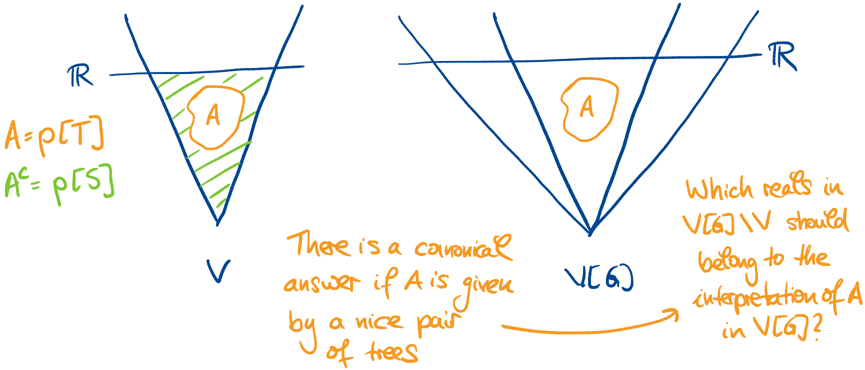

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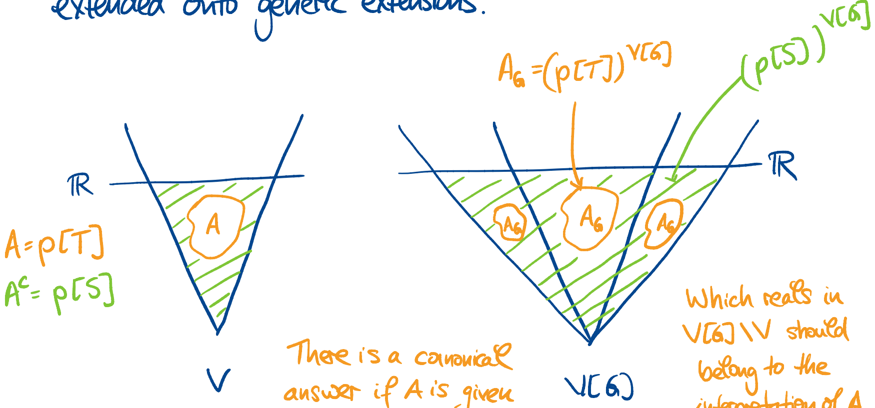

There is a canonical answer if A is given by a nice pair of trees

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Theorem (Larson-Sargsyan-Wilson, 2014)

Suppose there is a cardinal λ that is

- *a limit of Woodin cardinals, and*
- *a limit of (fully) strong cardinals.*

Then there is a model of

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Question (Sargsyan, 2013): Is this optimal?

Sargsyan's conjecture holds

Theorem (M, 2021)

Suppose there is a proper class model of

"$\mathrm{AD} +$ *all sets of reals are universally Baire*".

Then there is a transitive model $\mathcal{M}$ of ZFC containing all ordinals such that $\mathcal{M}$ has a cardinal λ that is

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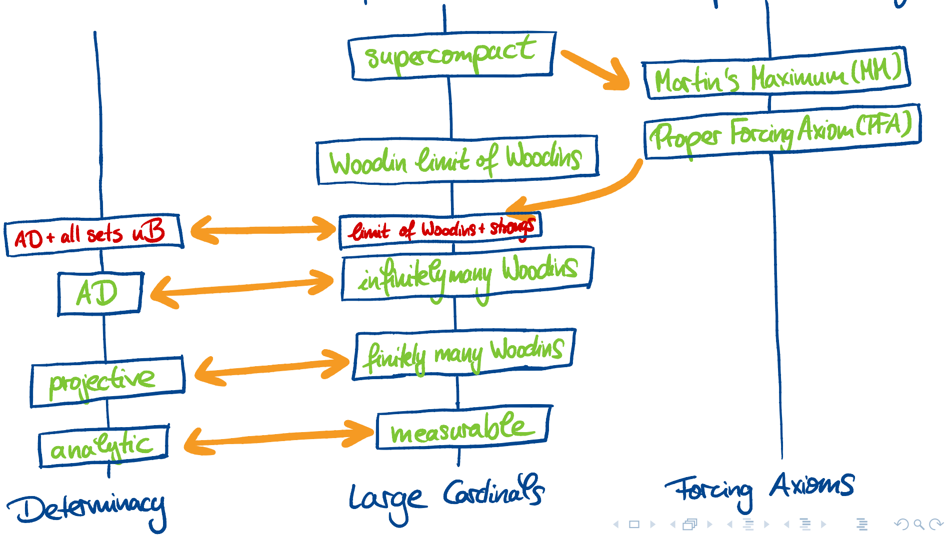

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What sort of canonical, generically absolute subsets of Γ^∞ can be added to the model $L(\Gamma^\infty, \mathbb{R})$?

Chang-type models

Possibilities for strong models of determinacy:

$L(\mathrm{Ord}^\omega)$

Chang model

Conjecture

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Gappo-M-Sargsyan, 2023 show AD in a special case here

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Theorem (Gappo-M-Sargsyan, 2023)

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a Chang-type model of "$\mathrm{AD}_\mathbb{R} + \Theta$ is regular $+\ \omega_1$ is $< \delta_\infty$-supercompact" for some $\delta_\infty > \Theta$.

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Suppose there is a proper class of Woodin cardinals that are limits of Woodin cardinals. Then there is a Chang-type model of
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What is the consistency strength of $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular $+\ \omega_1$ is supercompact?

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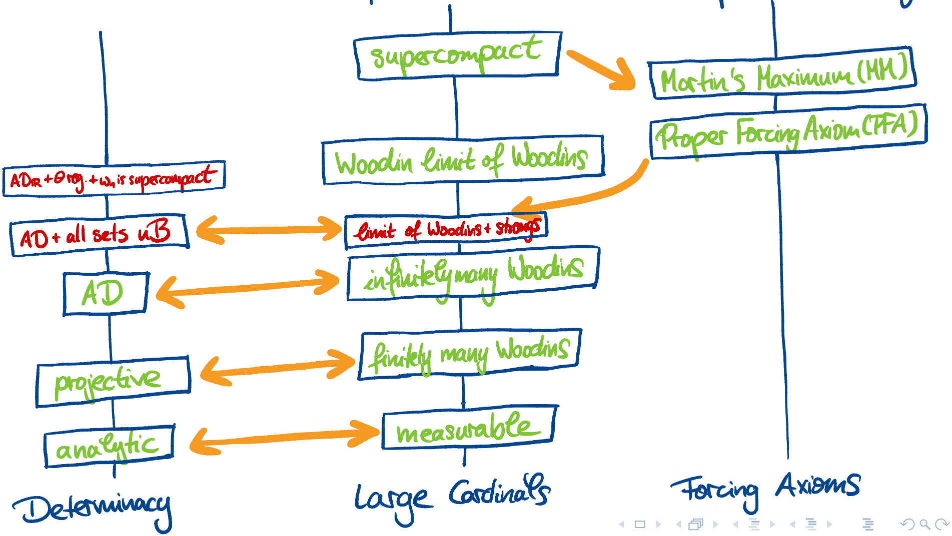

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These are (strong) models of determinacy.

They even satisfy a form of generic absoluteness.

Some things that can be proven

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Theorem (Steel, Woodin)

Suppose there is a proper class of Woodin cardinals. Let $V[g] \subseteq V[g * h]$ be set generic extensions of V. Then

1. $L(\mathbb{R}) \models \mathrm{AD}$ *and there is an elementary embedding*
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1. $L(\mathbb{R}) \models \mathrm{AD}$ *and there is an elementary embedding*
$$j \colon L(\mathbb{R}_g) \to L(\mathbb{R}_{g*h}),$$
2. *for any universally Baire set A, $L(A, \mathbb{R}) \models \mathrm{AD}$ and there is an elementary embedding*
$$j \colon L(A_g, \mathbb{R}_g) \to L(A_{g*h}, \mathbb{R}_{g*h}).$$

How about all uB sets?

For any g generic over V, write $\mathbb{R}_g = \mathbb{R}^{V[g]}$ and

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Sealing

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Definition (Woodin)

Sealing is the conjunction of the following statements.

1. For every set generic g over V, $L(\Gamma^\infty_g, \mathbb{R}_g) \models \mathrm{AD}^+$ and $\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.
2. For every set generic g over V and set generic h over $V[g]$, there is an elementary embedding

$$j \colon L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

Woodin's Sealing Theorem

Theorem (Woodin's Sealing Theorem)

Let κ be a supercompact cardinal and let g be $\mathrm{Col}(\omega, 2^{2^\kappa})$-generic over V. Suppose that there is a proper class of Woodin cardinals. Then Sealing *holds in $V[g]$.*

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Sargsyan-Trang showed that Sealing is consistent from a Woodin limit of Woodin cardinals.

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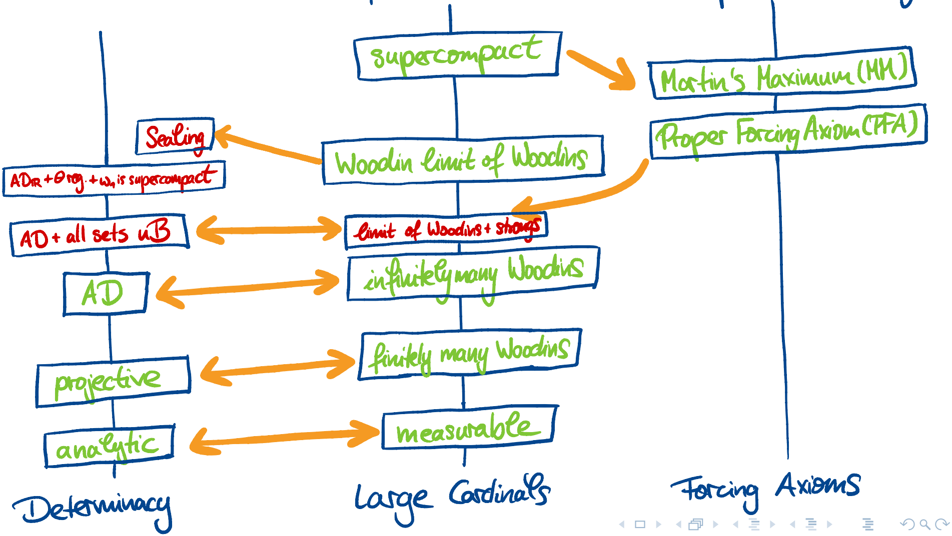

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- This is related to the study the union of all inner model theoretic operators over some X, e.g., $X^\#$, $M_1^\#(X)$, ...
- We would like to add the stack of all of these.

We consider the set of all canonical subsets of Γ^∞, call this $\mathcal{P}_{uB}(\Gamma^\infty) =: A_\infty$.

Determinacy for the uB-powerset

Write $\mathcal{A}_h^\infty = (\wp_{uB}(\Gamma^\infty))^{V[h]}$.

Theorem (M-Sargsyan, 2023)

Suppose κ is a supercompact cardinal, there is a proper class of inaccessible limits of Woodin cardinals and λ is an inaccessibe limit of Woodin cardinals above κ. Suppose $h \subseteq \mathrm{Col}(\omega, {<}\lambda)$ is V-generic. Then $L(\mathcal{A}_h^\infty) \models \mathsf{AD}^+$.

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We also prove a version of Sealing (generic absoluteness) for this model.

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The main barrier we are currently facing is a Woodin limit of Woodin cardinals.

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